UNIVERSITY OF TEXAS AT AUSTIN
Dept. of Electrical and Computer Engineering

Quiz #2

Date: April 4, 2001

Course: EE 313

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Last, First

- The exam is scheduled to last 75 minutes.
- Open books and open notes. You may refer to your homework and solution sets.
- Calculators are allowed.
- You may use any standalone computer system, i.e. one that is not connected to a network.
- All work should be performed on the quiz itself. If more space is needed, then use the backs of the pages.
- Fully justify your answers unless instructed otherwise.

Problem	Point Value	Your Score	Topic
1	25		Difference Equation
2	20		Step-To-Impulse Response
3	20		Tapped Delay Line
4	15		Sigma-Delta Modulation
5	20		Filters
Total	100		

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Problem 2.1 Differential Equation. 25 points.
Solve the following difference equation

$$y[k] - \frac{5}{6}y[k-1] + \frac{1}{6}y[k-2] = u[k]$$

with the initial conditions $y[-1] = 1$ and $y[-2] = 0$ by using the z-transform. Here, $u[k]$ is the discrete-time unit step function.

$$Y[z] - \frac{5}{6}\left(z^{-1}Y[z] + 1\right) + \frac{1}{6}\left(z^{-2}Y[z] + z^{-1}\right) = \frac{1}{1-z^{-1}}$$

$$Y[z] = \underbrace{\frac{1}{\left(1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}\right)\left(1-z^{-1}\right)}}_{\text{roots at } z=\frac{1}{2} \text{ and } z=\frac{1}{3}} + \frac{\frac{5}{6}-\frac{1}{6}z^{-1}}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}$$

$$\frac{1}{\left(1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}\right)\left(1-z^{-1}\right)} = \frac{-3}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-\frac{1}{3}z^{-1}} + \frac{3}{1-z^{-1}}$$

$$\frac{\frac{5}{6}-\frac{1}{6}z^{-1}}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}} = \frac{3\left(\frac{1}{2}\right)}{1-\frac{1}{2}z^{-1}} + \frac{-2\left(\frac{1}{3}\right)}{1-\frac{1}{3}z^{-1}}$$

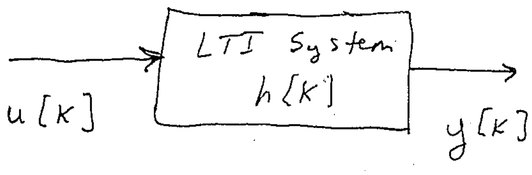
$$y[k] = -3\left(\frac{1}{2}\right)^k u[k] + \left(\frac{1}{3}\right)^k u[k] + 3u[k] + \frac{3}{2}\left(\frac{1}{2}\right)^k u[k] - \frac{2}{3}\left(\frac{1}{3}\right)^k u[k]$$

Problem 2.2 Step-To-Impulse Response. 20 points.

The step response of a discrete-time linear time-invariant (LTI) system is

$$\left(\frac{1-a^{k+1}}{1-a}\right) u[k]$$

where k is the discrete-time index and a is a constant such that $a \neq 1$. What is the impulse response of the LTI system?



$$y[k] = \left(\frac{1-a^{k+1}}{1-a}\right) u[k]$$

$$y[k] = h[k] * u[k]$$

Approach #1 : Discrete-time convolution table, Lathi p. 217:

$$h[k] = a^k u[k]$$

Approach #2 : $Y[z] = H[z]\, U[z] \Rightarrow H[z] = \frac{Y[z]}{U[z]}$

$$Y[z] = \frac{1}{1-a}\left[\frac{1}{1-z^{-1}} - \frac{a}{1-az^{-1}}\right]; \quad U[z] = \frac{1}{1-z^{-1}}$$

$$\frac{Y[z]}{U[z]} = \frac{1}{1-a}\left[1 - a\,\frac{1-z^{-1}}{1-az^{-1}}\right] = H[z]$$

$$\frac{Y(z)}{U[z]} = \frac{1}{1-a}\left[\frac{(1-az^{-1}) + (-a+az^{-1})}{1-az^{-1}}\right] = H[z]$$

$$\frac{Y[z]}{U[z]} = \frac{1}{1-a}\left[\frac{1-a}{1-az^{-1}}\right] = H[z]$$

$$H[z] = \frac{1}{1-az^{-1}} \Rightarrow h[k] = a^k u[k]$$

Problem 2.3 Tapped Delay Line. 20 points.

For input $x(t)$, the output $y(t)$ of a continuous-time tapped delay line is

$$y(t) = \sum_{m=0}^{M-1} a_m \, x(t - mT)$$

where T is delay between taps and $M - 1$ is the number of delay elements.

(a) Compute the transfer function $H(s)$. 9 points.

$$Y(s) = \sum_{m=0}^{M-1} a_m X(s) e^{-mTs} \Rightarrow \frac{Y(s)}{X(s)} = \sum_{m=0}^{M-1} a_m e^{-mTs} = H(s)$$

i. How many poles are in the transfer function? 0

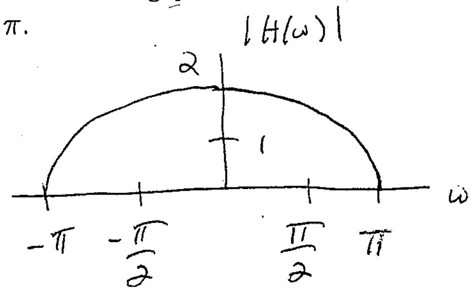
ii. How many zeros are in the transfer function? $M-1$

(b) Compute the frequency response $H(\omega)$, where ω is in units of rad/s. 5 points.

$$H(\omega) = H(s)\Big|_{s=j\omega} = \sum_{m=0}^{M-1} a_m e^{-jmT\omega}$$

(c) For each of the following sets of coefficients, state whether the filter is lowpass, highpass, bandpass, or bandstop. You can let $T = 1$ s. In determining your answer, consider only the following range of frequencies: $-\pi \le w \le \pi$.

i. $M = 2$, $a_0 = 1$, and $a_1 = 1$. 3 points.



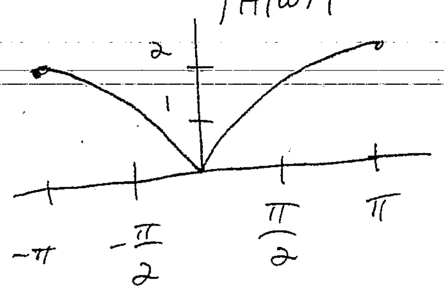
$$H(\omega) = 1 + e^{-j\omega}$$

Lowpass

$$|H(0)| = 2 \qquad |H(\pi)| = |H(-\pi)| = 0$$

$$|H(\tfrac{\pi}{2})| = |H(-\tfrac{\pi}{2})| = \sqrt{2}$$

ii. $M = 2$, $a_0 = 1$, and $a_1 = -1$. 3 points.



$$H(\omega) = 1 - e^{-j\omega}$$

Highpass

$$|H(0)| = 0 \quad |H(\pi)| = |H(-\pi)| = 2$$

$$|H(\tfrac{\pi}{2})| = |H(-\tfrac{\pi}{2})| = \sqrt{2}$$

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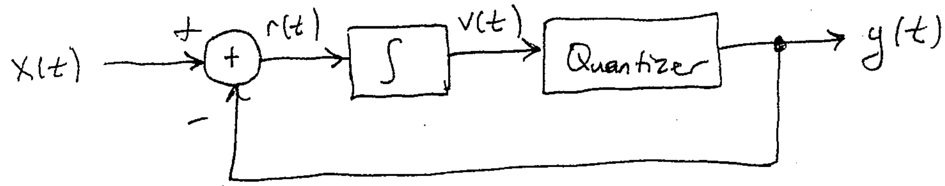
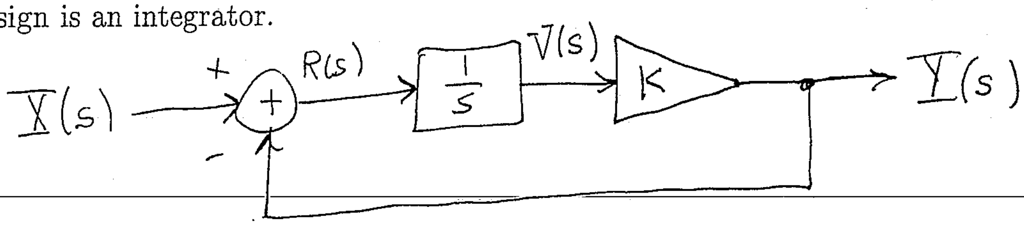
Problem 2.4 Sigma-Delta Modulation. 15 points.

Shown below is a type of continuous-time sigma-delta modulator:



We can approximate the effect of the quantizer as a gain K, which would make the overall system linear and time-invariant. Replace the quantizer with a gain of K and derive the transfer function from input $x(t)$ to output $y(t)$. The LTI system shown graphically as an integral sign is an integrator.



$$R(s) = X(s) - Y(s)$$

$$V(s) = \frac{1}{s} R(s)$$

$$Y(s) = K\, V(s)$$

Combining these three equations:

$$Y(s) = K \cdot \frac{1}{s} \cdot \left(X(s) - Y(s) \right)$$

$$\left(1 + \frac{K}{s}\right) Y(s) = \frac{K}{s} X(s)$$

$$\frac{Y(s)}{X(s)} = \frac{\frac{K}{s}}{1 + \frac{K}{s}} = \frac{K}{s + K}$$

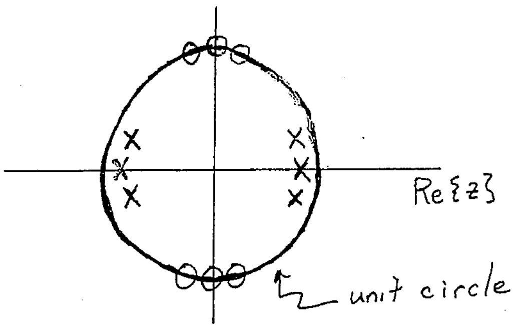
Problem 2.5 Filters. 20 points.

Shown below are the pole-zero diagrams of four different transfer functions. For each transfer function, the X's indicate locations of poles and O's indicate locations of zeros. For each transfer function, indicate the type of filter (lowpass, highpass, bandpass, bandstop, or allpass) and give one application of that type of filter (e.g. to speech, audio, or image processing, or communications systems).

– Poles near the unit circle indicate the passband

– Zeros on the unit circle indicate the stopband

(a)



Z domain

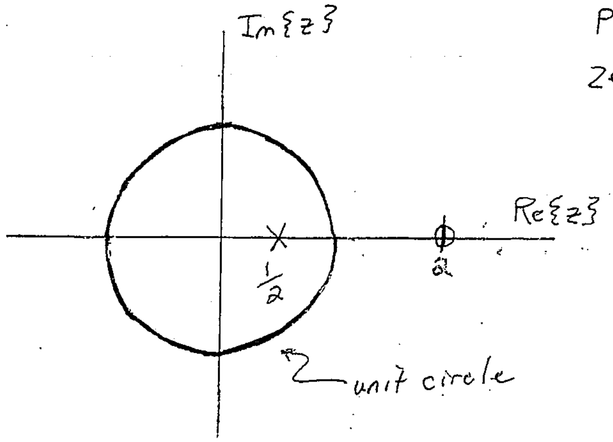
Bandstop

– passbands at $z=1$ ($\omega=0$) and $z=-1$ ($\omega=\pi$ and $\omega=-\pi$)

– stopbands at $z=j$ ($\omega=\frac{\pi}{2}$) and $z=-j$ ($\omega=-\frac{\pi}{2}$)

Application – Communication systems to reject in-band interference, e.g. ham radio band in VDSL transmission

(b)



Z domain

pole at $z=\frac{1}{2}$

zero at $z=2$

$$H(z) = b_m \frac{z-2}{z-\frac{1}{2}}$$

Evaluate points on unit circle ($b_m=1$):

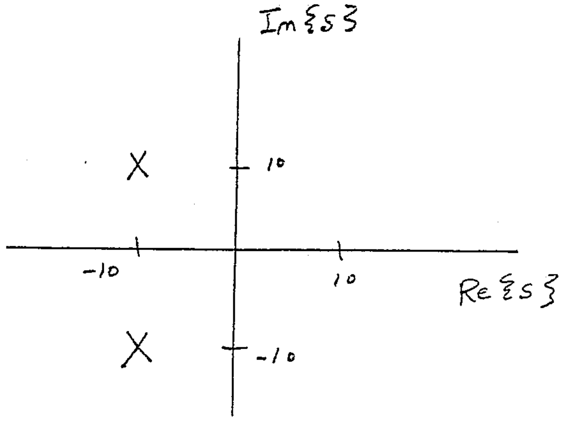
$|H(1)| = 2$ $\quad |H(j)| = 2$

$|H(-1)| = 2$ $\quad |H(-j)| = 2$

Allpass

Application – Phase correction, e.g. after analog-to-digital conversion (audio)

(c)



Laplace domain

Lowpass

poles at $s = -10 \pm j10$.

no zeros.

$$H(s) = \frac{1}{(s+10-j10)(s+10+j10)}$$

Evaluate various values of s

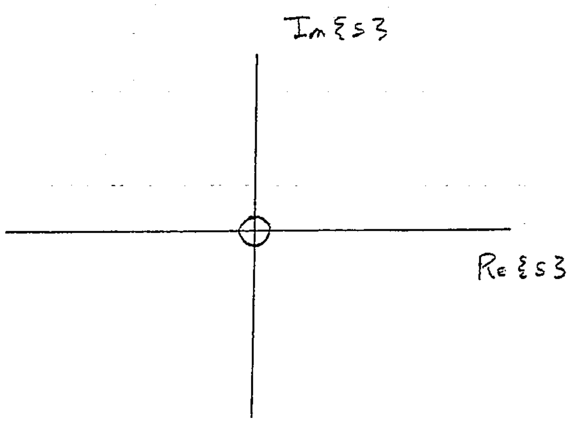
$$|H(0)| = \frac{1}{100}$$

$$|H(10)| = \left|\frac{1}{(10)(10+j20)}\right| = \left|\frac{1}{100+j200}\right|$$

$$|H(-10)| = \left|\frac{1}{(10-j20)(10)}\right| = \left|\frac{1}{100-j200}\right|$$

Application — Lowpass filtering before analog-to-digital conversion to enforce sampling theorem

(d)



Laplace domain —

no poles.

zero at $s=0$

This is a differentiator:

$$|H(\omega)| = |j\omega| = |\omega|$$

Highpass.

Application — Enhancing edges and texture in images.